Initial forms of stable invariants for additive group actions

Shigeru Kuroda*

Abstract

The Derksen–Hadas–Makar-Limanov theorem (2001) says that the invariants for nontrivial actions of the additive group on a polynomial ring have no intruder. In this paper, we generalize this theorem to the case of stable invariants.

1 Introduction

Throughout this paper, let k be a domain unless otherwise stated, and $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k, where $n \geq 1$. For each

$$f = \sum_{i_1,\ldots,i_n} u_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in k[\mathbf{x}] \tag{1.1}$$

with $u_{i_1,\ldots,i_n} \in k$, we define $\operatorname{supp}(f) = \{(i_1, \ldots, i_n) \mid u_{i_1,\ldots,i_n} \neq 0\}$. We call the convex hull $\operatorname{New}(f)$ of $\operatorname{supp}(f)$ in $\mathbf{R}^n$ the *Newton polytope* of f. A vertex $(i_1, \ldots, i_n)$ of $\operatorname{New}(f)$ is called an *intruder* of f if $i_l \neq 0$ for $l = 1, \ldots, n$.

Let $\mathbf{G}_{\mathrm{a}} = \operatorname{Spec} k[z]$ be the additive group, where z is an indeterminate. A homomorphism $\sigma : k[\mathbf{x}] \to k[\mathbf{x}][z] = k[\mathbf{x}] \otimes_k k[z]$ of k-algebras is called a $\mathbf{G}_{\mathrm{a}}$-*action* on $k[\mathbf{x}]$ if $\pi \circ \sigma = \mathrm{id}_{k[\mathbf{x}]}$, and the diagram

$$\begin{tikzcd}
k[\mathbf{x}] \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & k[\mathbf{x}] \otimes_k k[z] \arrow[d, "\sigma \otimes \mathrm{id}_{k[z]}"] \\
k[\mathbf{x}] \otimes_k k[z] \arrow[r, "\mathrm{id}_{k[\mathbf{x}]} \otimes \mu"'] & k[\mathbf{x}] \otimes_k k[z] \otimes_k k[z]
\end{tikzcd}$$

2010 *Mathematical Subject Classification*. Primary 14L30; Secondary 14R10.

*Partly supported by the Grant-in-Aid for Young Scientists (B) 24740022, Japan Society for the Promotion of Science.

commutes. Here, $\pi : k[\mathbf{x}][z] \to k[\mathbf{x}]$ and $\mu : k[z] \to k[z] \otimes_k k[z]$ are the homomorphisms of $k[\mathbf{x}]$-algebras and k-algebras defined by $\pi(z) = 0$ and $\mu(z) = z \otimes 1 + 1 \otimes z$, respectively. We note that $\sigma(f) = f$ if and only if $\sigma(f)$ belongs to $k[\mathbf{x}]$ for $f \in k[\mathbf{x}]$. When this is the case, we call f an *invariant* for σ. The set $k[\mathbf{x}]^{\mathbf{G}_\mathrm{a}} := \sigma^{-1}(k[\mathbf{x}])$ of invariants for σ forms a k-subalgebra of $k[\mathbf{x}]$. The $\mathbf{G}_\mathrm{a}$-action defined by the inclusion map $k[\mathbf{x}] \to k[\mathbf{x}][z]$ is called a *trivial* $\mathbf{G}_\mathrm{a}$-action. A $\mathbf{G}_\mathrm{a}$-action is trivial if and only if $k[\mathbf{x}]^{\mathbf{G}_\mathrm{a}} = k[\mathbf{x}]$.

The Derksen–Hadas–Makar-Limanov theorem [DHM, Theorem 3.1] says that the invariants for any nontrivial $\mathbf{G}_\mathrm{a}$-action on $k[\mathbf{x}]$ have no intruder. This theorem implies that, if $f_1, \ldots, f_n \in k[\mathbf{x}]$ satisfy $k[f_1, \ldots, f_n] = k[\mathbf{x}]$, then no element of $k[f_2, \ldots, f_n]$ has an intruder [DHM, Corollary 3.2].

The purpose of this paper is to present "stable versions" of the results above. For $m \geq n$, let $k[\overline{\mathbf{x}}] = k[x_1, \ldots, x_m]$ be the polynomial ring in m variables over k. We call $f \in k[\mathbf{x}]$ a *stable* $\mathbf{G}_\mathrm{a}$-*invariant* of $k[\mathbf{x}]$ if there exist $m \geq n$ and a $\mathbf{G}_\mathrm{a}$-action on $k[\overline{\mathbf{x}}]$ for which $k[\overline{\mathbf{x}}]^{\mathbf{G}_\mathrm{a}}$ contains f, but does not contain $k[\mathbf{x}]$. If $f \in k[\mathbf{x}]$ is an invariant for some nontrivial $\mathbf{G}_\mathrm{a}$-action on $k[\mathbf{x}]$, then f is a stable $\mathbf{G}_\mathrm{a}$-invariant by definition. However, it is not known whether the converse holds in general (cf. Section 3).

We generalize the Derksen–Hadas–Makar-Limanov theorem as follows.

Theorem 1.1. *No stable* $\mathbf{G}_\mathrm{a}$*-invariant of* $k[\mathbf{x}]$ *has an intruder.*

This theorem is a consequence of a more general result as follows. Let Γ be a *totally ordered additive group*, i.e., an additive group equipped with a total ordering such that $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$ for each $\alpha, \beta, \gamma \in \Gamma$. For example, $\mathbf{R}$ is a totally ordered additive group for the standard ordering. Take any $\mathbf{w} = (w_1, \ldots, w_n) \in \Gamma^n$. We denote $a \cdot \mathbf{w} = a_1 w_1 + \cdots + a_n w_n$ for $a = (a_1, \ldots, a_n) \in \mathbf{Z}^n$. For each $f \in k[\mathbf{x}] \setminus \{0\}$, we define the $\mathbf{w}$-*degree* $\deg_\mathbf{w} f$ and $\mathbf{w}$-*initial form* $f^\mathbf{w}$ by

$$\deg_\mathbf{w} f = \max\{a \cdot \mathbf{w} \mid a \in \operatorname{supp}(f)\} \quad \text{and} \quad f^\mathbf{w} = \sum_{i_1, \ldots, i_n} u'_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $u'_{i_1, \ldots, i_n} = u_{i_1, \ldots, i_n}$ if $(i_1, \ldots, i_n) \cdot \mathbf{w} = \deg_\mathbf{w} f$, and $u'_{i_1, \ldots, i_n} = 0$ otherwise. When $f = 0$, we define $\deg_\mathbf{w} f = -\infty$ and $f^\mathbf{w} = 0$. Then, it holds that

$$\deg_\mathbf{w} fg = \deg_\mathbf{w} f + \deg_\mathbf{w} g \quad \text{and} \quad (fg)^\mathbf{w} = f^\mathbf{w} g^\mathbf{w} \tag{1.2}$$

for each $f, g \in k[\mathbf{x}]$. We remark that $f \in k[\mathbf{x}] \setminus \{0\}$ has no intruder if and only if, for each $\mathbf{w} \in \mathbf{R}^n$ with $f^\mathbf{w}$ a monomial, there exists $1 \leq i \leq n$ such that x_i does not divide $f^\mathbf{w}$.

Now, let $\phi : k[\mathbf{x}] \to k[\overline{\mathbf{x}}][z]$ be a homomorphism of k-algebras such that $\phi(k[\mathbf{x}])$ is not contained in $k[\overline{\mathbf{x}}]$, and $\mathbf{w}$ an element of Γ^n which satisfies the following condition, where $p_i(z) := \phi(x_i) \in k[\overline{\mathbf{x}}][z]$ for each i:

$(*)$ There exists $\mathbf{v} \in \Gamma^m$ such that $p_1(0)^{\mathbf{v}}, \ldots, p_n(0)^{\mathbf{v}}$ are algebraically independent over k, and $\deg_{\mathbf{v}} p_i(0) = w_i$ for $i = 1, \ldots, n$.

In this situation, $k[\mathbf{x}]^\phi := \phi^{-1}(k[\overline{\mathbf{x}}])$ is a proper k-subalgebra of $k[\mathbf{x}]$. The following theorem will be proved in the next section.

Theorem 1.2. *Let $\phi : k[\mathbf{x}] \to k[\overline{\mathbf{x}}][z]$ and $\mathbf{w} \in \Gamma^n$ be as above, and $S \subset k[\mathbf{x}] \setminus \{0\}$ such that* $\text{trans.deg}_k\, k[S] = n$*. Then, there exists $g \in S$ such that g does not divide $f^{\mathbf{w}}$ for any $f \in k[\mathbf{x}]^\phi \setminus \{0\}$.*

If f is a stable $\mathbf{G}_{\mathrm{a}}$-invariant of $k[\mathbf{x}]$, then there exist $m \geq n$ and a $\mathbf{G}_{\mathrm{a}}$-action $\sigma : k[\overline{\mathbf{x}}] \to k[\overline{\mathbf{x}}][z]$ such that $\sigma^{-1}(k[\overline{\mathbf{x}}])$ contains f, but does not contain $k[\mathbf{x}]$. We define $\phi = \sigma|_{k[\mathbf{x}]}$. Then, $\phi(k[\mathbf{x}]) = \sigma(k[\mathbf{x}])$ is not contained in $k[\overline{\mathbf{x}}]$. We claim that any $\mathbf{w} \in \Gamma^n$ satisfies $(*)$ for this ϕ. In fact, since $\pi \circ \sigma = \mathrm{id}_{k[\overline{\mathbf{x}}]}$, we have $p_i(0) = \pi(\sigma(x_i)) = x_i$ for each i. Hence, $(*)$ holds for $\mathbf{v} = (\mathbf{w}, 0, \ldots, 0) \in \Gamma^m$. Clearly, f belongs to $\sigma^{-1}(k[\overline{\mathbf{x}}]) \cap k[\mathbf{x}] = \phi^{-1}(k[\overline{\mathbf{x}}]) = k[\mathbf{x}]^\phi$. Therefore, we obtain the following theorem as a consequence of Theorem 1.2.

Theorem 1.3. *Let f be a nonzero stable $\mathbf{G}_{\mathrm{a}}$-invariant of $k[\mathbf{x}]$, and $S \subset k[\mathbf{x}] \setminus \{0\}$ such that* $\text{trans.deg}_k\, k[S] = n$*. Then, $f^{\mathbf{w}}$ is not divisible by an element of S for each $\mathbf{w} \in \Gamma^n$.*

Since $S = \{x_1, \ldots, x_n\}$ satisfies $\text{trans.deg}_k\, k[S] = n$, Theorem 1.1 follows from Theorem 1.3 by virtue of the above remark on intruders. As another application of Theorem 1.3, we obtain the following theorem.

Theorem 1.4. *Let $m \geq n$ and $f_1, \ldots, f_m \in k[\overline{\mathbf{x}}]$ be such that $k[f_1, \ldots, f_m] = k[\overline{\mathbf{x}}]$ and $k[\mathbf{x}]$ is not contained in $k[f_2, \ldots, f_m]$, and let $S \subset k[\mathbf{x}] \setminus \{0\}$ be such that* $\text{trans.deg}_k\, k[S] = n$*. Then, for each $\mathbf{w} \in \Gamma^n$, there exists $g \in S$ such that g does not divide $f^{\mathbf{w}}$ for any $f \in k[f_2, \ldots, f_m] \cap k[\mathbf{x}] \setminus \{0\}$.*

In fact, we have $k[\overline{\mathbf{x}}]^{\mathbf{G}_{\mathrm{a}}} = k[f_2, \ldots, f_m]$ for the $\mathbf{G}_{\mathrm{a}}$-action on $k[\overline{\mathbf{x}}]$ defined by $f_1 \mapsto f_1 + z$ and $f_i \mapsto f_i$ for each $i \geq 2$. Hence, every element of $k[f_2, \ldots, f_m] \cap k[\mathbf{x}]$ is a stable $\mathbf{G}_{\mathrm{a}}$-invariant of $k[\mathbf{x}]$ unless $k[\mathbf{x}]$ is contained in $k[f_2, \ldots, f_m]$.

We call $f \in k[\mathbf{x}]$ a *coordinate* of $k[\mathbf{x}]$ if there exist $f_2, \ldots, f_n \in k[\mathbf{x}]$ such that $k[f, f_2, \ldots, f_n] = k[\mathbf{x}]$, and a *stable coordinate* of $k[\mathbf{x}]$ if there exists $m \geq n$ such that f is a coordinate of $k[\overline{\mathbf{x}}]$. By definition, every coordinate of $k[\mathbf{x}]$ is a stable coordinate of $k[\mathbf{x}]$. Since k is a domain, the converse is clear if $n = 1$. If $n = 2$, however, not every stable coordinate of $k[\mathbf{x}]$ is a coordinate of $k[\mathbf{x}]$ (cf. [BD]; see also Section 3).

Assume that $n \geq 2$, and let f_1 be a stable coordinate of $k[\mathbf{x}]$. Then, there exist $m \geq n$ and $f_2, \ldots, f_m \in k[\overline{\mathbf{x}}]$ such that $k[f_1, \ldots, f_m] = k[\overline{\mathbf{x}}]$. Since

$n \geq 2$, we see that $k[\mathbf{x}]$ is not contained in $k[f_1] = \bigcap_{i=2}^{m} k[\{f_j \mid j \neq i\}]$. Hence, there exists $2 \leq i_0 \leq m$ such that $k[\mathbf{x}]$ is not contained in $k[\{f_j \mid j \neq i_0\}]$. Since f_1 belongs to $k[\{f_j \mid j \neq i_0\}] \cap k[\mathbf{x}] \setminus \{0\}$, we obtain the following corollary to Theorem 1.4.

Corollary 1.5. *Assume that $n \geq 2$. Let f be a stable coordinate of $k[\mathbf{x}]$, and let $S \subset k[\mathbf{x}] \setminus \{0\}$ be such that* $\operatorname{trans.deg}_k k[S] = n$. *Then, $f^{\mathbf{w}}$ is not divisible by an element of S for each $\mathbf{w} \in \Gamma^n$.*

It is possible that an element of $k[\mathbf{x}]$ which is not a coordinate of $k[\mathbf{x}]$ becomes a coordinate of $k_0[\mathbf{x}]$, where k_0 is the field of fractions of k. Hence, an element of $k[\mathbf{x}]$ which is not a stable coordinate of $k[\mathbf{x}]$ can be a stable coordinate of $k_0[\mathbf{x}]$. We note that the conclusion of Corollary 1.5 holds if only $f \in k[\mathbf{x}]$ is a stable coordinate of $k_0[\mathbf{x}]$. Similarly, the conclusion of Theorem 1.3 holds if only $f \in k[\mathbf{x}] \setminus \{0\}$ is a stable $\mathbf{G}_{\mathrm{a}}$-invariant of $k_0[\mathbf{x}]$.

The author would like to thank Prof. Hideo Kojima for pointing out Freudenburg's paper mentioned in Section 3, and Prof. Amartya K. Dutta and Prof. Neena Gupta for the invaluable remarks in Section 3.

2 Proof of Theorem 1.2

For a domain R and a subring S of R, we say that S is *factorially closed* in R if $ab \in S$ implies $a, b \in S$ for each $a, b \in R \setminus \{0\}$. We remark that R is algebraically closed and factorially closed in the polynomial ring $R[x_1, \ldots, x_n]$ if R is a domain.

Let R and R' be domains, and $\psi : R \to R'[x_1, \ldots, x_n]$ a homomorphism of rings. Then, the following lemma holds for $R^{\psi} := \psi^{-1}(R')$.

Lemma 2.1. *If ψ is injective, then R^{ψ} is algebraically closed and factorially closed in R.*

Proof. Since R' is algebraically closed and factorially closed in $R'[x_1, \ldots, x_n]$, we see that $\psi(R) \cap R'$ is algebraically closed and factorially closed in $\psi(R)$. Hence, the lemma follows by the injectivity of ψ. □

For $\mathbf{w} \in \Gamma^n$ and $(0, \ldots, 0) \neq (f_1, \ldots, f_l) \in k[\mathbf{x}]^l$ with $l \geq 1$, we set

$$\delta = \max\{\deg_{\mathbf{w}} f_i \mid i = 1, \ldots, l\} \quad \text{and} \quad I = \{i \mid \deg_{\mathbf{w}} f_i = \delta\}.$$

Then, we have the following lemma, which can be verified easily.

Lemma 2.2. *If $\sum_{i \in I} f_i^{\mathbf{w}} \neq 0$, then we have $(f_1 + \cdots + f_l)^{\mathbf{w}} = \sum_{i \in I} f_i^{\mathbf{w}}$.*

Now, let $\psi : k[\mathbf{x}] \to k[\overline{\mathbf{x}}]$ be a homomorphism of k-algebras with $\psi(x_i) \neq 0$ for each i. For $\mathbf{u} \in \Gamma^m$, we define a homomorphism $\psi^{\mathbf{u}} : k[\mathbf{x}] \to k[\overline{\mathbf{x}}]$ of k-algebras by $\psi^{\mathbf{u}}(x_i) = \psi(x_i)^{\mathbf{u}}$ for $i = 1, \ldots, n$. Set

$$\mathbf{u}_\psi = (\deg_{\mathbf{u}} \psi(x_1), \ldots, \deg_{\mathbf{u}} \psi(x_n)) \in \Gamma^n.$$

With this notation, the following proposition holds.

Proposition 2.3. *If* $\psi^{\mathbf{u}}(f^{\mathbf{u}_\psi}) \neq 0$ *for* $f \in k[\mathbf{x}]$, *then we have* $\psi(f)^{\mathbf{u}} = \psi^{\mathbf{u}}(f^{\mathbf{u}_\psi})$.

Proof. Write f as in (1.1), and set $f_i = u_i x_1^{i_1} \cdots x_n^{i_n}$ for each $i = (i_1, \ldots, i_n)$. Then, we have $f = \sum_i f_i$ and $\psi(f) = \sum_i \psi(f_i)$. We apply Lemma 2.2 to $(\psi(f_i))_{i \in \mathrm{supp}(f)}$. Note that $\deg_{\mathbf{u}} \psi(f_i) = i \cdot \mathbf{u}_\psi$ and $\psi(f_i)^{\mathbf{u}} = \psi^{\mathbf{u}}(f_i)$ for each $i \in \mathrm{supp}(f)$ by (1.2). Hence, we get

$$\delta = \max\{\deg_{\mathbf{u}} \psi(f_i) \mid i \in \mathrm{supp}(f)\} = \max\{i \cdot \mathbf{u}_\psi \mid i \in \mathrm{supp}(f)\} = \deg_{\mathbf{u}_\psi} f,$$

and so

$$I = \{i \mid \deg_{\mathbf{u}} \psi(f_i) = \delta\} = \{i \mid i \cdot \mathbf{u}_\psi = \deg_{\mathbf{u}_\psi} f\}.$$

Thus, we have $\sum_{i \in I} f_i = f^{\mathbf{u}_\psi}$. Therefore, we know that

$$\sum_{i \in I} \psi(f_i)^{\mathbf{u}} = \sum_{i \in I} \psi^{\mathbf{u}}(f_i) = \psi^{\mathbf{u}} \left(\sum_{i \in I} f_i \right) = \psi^{\mathbf{u}}(f^{\mathbf{u}_\psi}) \neq 0.$$

By Lemma 2.2, it follows that $\psi(f)^{\mathbf{u}} = \left(\sum_i \psi(f_i)\right)^{\mathbf{u}}$ is equal to the left-hand side of the preceding equality, and hence to $\psi^{\mathbf{u}}(f^{\mathbf{u}_\psi})$. $\square$

Fix any $1 \leq l \leq m$. Let $k[\mathbf{x}]^\psi$ and $k[\mathbf{x}]^{\psi^{\mathbf{u}}}$ be the k-subalgebras of $k[\mathbf{x}]$ defined as the inverse images of $k[x_1, \ldots, x_l]$ by ψ and $\psi^{\mathbf{u}}$, respectively. Then, we have the following theorem.

Theorem 2.4. (i) $f^{\mathbf{u}_\psi}$ *belongs to* $k[\mathbf{x}]^{\psi^{\mathbf{u}}}$ *for each* $f \in k[\mathbf{x}]^\psi$.
(ii) *Let* $S \subset k[\mathbf{x}] \setminus \{0\}$ *be such that* $\mathrm{trans.deg}_k\, k[S] = n$. *If* $\psi^{\mathbf{u}}$ *is injective and* $\psi^{\mathbf{u}}(k[\mathbf{x}])$ *is not contained in* $k[x_1, \ldots, x_l]$, *then there exists* $g \in S$ *such that* $f^{\mathbf{u}_\psi}$ *is not divisible by* g *for any* $f \in k[\mathbf{x}]^\psi \setminus \{0\}$.

Proof. (i) Since $k[\mathbf{x}]^{\psi^{\mathbf{u}}}$ is the inverse image of $k[x_1, \ldots, x_l]$ by $\psi^{\mathbf{u}}$, it suffices to check that $\psi^{\mathbf{u}}(f^{\mathbf{u}_\psi})$ belongs to $k[x_1, \ldots, x_l]$. This is clear if $\psi^{\mathbf{u}}(f^{\mathbf{u}_\psi}) = 0$. If $\psi^{\mathbf{u}}(f^{\mathbf{u}_\psi}) \neq 0$, then we have $\psi^{\mathbf{u}}(f^{\mathbf{u}_\psi}) = \psi(f)^{\mathbf{u}}$ by Proposition 2.3. This is an element of $k[x_1, \ldots, x_l]$, since so is $\psi(f)$ by the choice of f.

(ii) Since $\psi^{\mathbf{u}}$ is injective by assumption, $k[\mathbf{x}]^{\psi^{\mathbf{u}}}$ is algebraically closed and factorially closed in $k[\mathbf{x}]$ by Lemma 2.1. Since $\psi^{\mathbf{u}}(k[\mathbf{x}])$ is not contained in

$k[x_1, \ldots, x_l]$, we have $k[\mathbf{x}]^{\psi^{\mathbf{u}}} \neq k[\mathbf{x}]$. Hence, $\text{trans.deg}_k\, k[\mathbf{x}]^{\psi^{\mathbf{u}}}$ is less than n. Since $\text{trans.deg}_k\, k[S] = n$ by assumption, we may find $y_1, \ldots, y_n \in S$ such that $\text{trans.deg}_k\, k[y_1, \ldots, y_n] = n$. Suppose that the assertion is false. Then, for each $1 \leq i \leq n$, there exists $f_i \in k[\mathbf{x}]^{\psi} \setminus \{0\}$ such that $f_i^{\mathbf{u}_\psi}$ is divisible by y_i. Then, $(f_1 \cdots f_n)^{\mathbf{u}_\psi}$ belongs to $k[\mathbf{x}]^{\psi^{\mathbf{u}}}$ by (i), and is divisible by $y_1, \ldots, y_n$ due to (1.2). Since $k[\mathbf{x}]^{\psi^{\mathbf{u}}}$ is factorially closed in $k[\mathbf{x}]$, it follows that $y_1, \ldots, y_n$ belong to $k[\mathbf{x}]^{\psi^{\mathbf{u}}}$, a contradiction. $\square$

Now, let us complete the proof of Theorem 1.2. Note that Γ is torsion-free due to the structure of total ordering. Hence, we may regard Γ as a subgroup of $\mathbf{Q} \otimes_{\mathbf{Z}} \Gamma$ which also has a structure of totally ordered additive group induced from Γ. Write

$$\phi(x_i) = p_i(z) = \sum_{j \geq 0} p_{i,j} z^j$$

for $i = 1, \ldots, n$, where $p_{i,j} \in k[\overline{\mathbf{x}}]$ for each j. Since $\phi(k[\mathbf{x}])$ is not contained in $k[\overline{\mathbf{x}}]$ by assumption, we have $p_{i,j} \neq 0$ for some $1 \leq i \leq n$ and $j \geq 1$. Following [DHM], we define $\mathbf{u} = (\mathbf{v}, -\deg_{\mathbf{v}} \phi) \in (\mathbf{Q} \otimes_{\mathbf{Z}} \Gamma)^{m+1}$, where

$$\deg_{\mathbf{v}} \phi := \max \left\{ \frac{1}{j} \left(\deg_{\mathbf{v}} p_{i,j} - w_i \right) \;\middle|\; i = 1, \ldots, n,\ j \geq 1 \right\}.$$

We show that $\mathbf{u}_\phi = \mathbf{w}$, $\phi^{\mathbf{u}}$ is injective, and $\phi^{\mathbf{u}}(k[\mathbf{x}])$ is not contained in $k[\overline{\mathbf{x}}]$. Then, the proof is completed by Theorem 2.4 (ii).

By the maximality of $\deg_{\mathbf{v}} \phi$, we have $\deg_{\mathbf{v}} p_{i,j} - w_i \leq j \deg_{\mathbf{v}} \phi$, and so

$$\deg_{\mathbf{u}} p_{i,j} z^j = \deg_{\mathbf{v}} p_{i,j} - j \deg_{\mathbf{v}} \phi \leq w_i$$

for each $1 \leq i \leq n$ and $j \geq 1$. Since $\deg_{\mathbf{u}} p_{i,0} z^0 = \deg_{\mathbf{v}} p_{i,0} = \deg_{\mathbf{v}} p_i(0) = w_i$, it follows that $\deg_{\mathbf{u}} \phi(x_i) = \max\{\deg_{\mathbf{u}} p_{i,j} z^j \mid j \geq 0\} = w_i$ for each i. This proves that $\mathbf{u}_\phi = \mathbf{w}$. Let $J(i)$ be the set of $j \geq 1$ such that $\deg_{\mathbf{u}} p_{i,j} z^j = w_i$ for each i. Then, we have

$$\phi^{\mathbf{u}}(x_i) = \phi(x_i)^{\mathbf{u}} = \sum_{j \in J(i)} p_{i,j}^{\mathbf{v}} z^j + p_i(0)^{\mathbf{v}}.$$

Since $p_i(0)^{\mathbf{v}}$'s are algebraically independent over k, we see that $\phi^{\mathbf{u}}(x_i)$'s are algebraically independent over k. Therefore, $\phi^{\mathbf{u}}$ is injective. By the definition of $\deg_{\mathbf{v}} \phi$, there exist $1 \leq i_0 \leq n$ and $j_0 \geq 1$ such that $\deg_{\mathbf{v}} p_{i_0,j_0} - w_{i_0} = j_0 \deg_{\mathbf{v}} \phi$. Then, we have $\deg_{\mathbf{u}} p_{i_0,j_0} z^{j_0} = w_{i_0}$. Hence, the monomial z^{j_0} appears in $\phi^{\mathbf{u}}(x_{i_0})$ with coefficient $p_{i_0,j_0}^{\mathbf{v}} \neq 0$. Thus, $\phi^{\mathbf{u}}(x_{i_0})$ does not belong to $k[\overline{\mathbf{x}}]$. Therefore, $\phi^{\mathbf{u}}(k[\mathbf{x}])$ is not contained in $k[\overline{\mathbf{x}}]$. This completes the proof of Theorem 1.2.

3 Remarks on stable coordinates and stable $\mathbf{G}_{\mathrm{a}}$-invariants

Shpilrain-Yu [SY] remarked that every stable coordinate of $\mathbf{C}[\mathbf{x}]$ is a coordinate of $\mathbf{C}[\mathbf{x}]$ for $n = 2, 3$. In the case of $n = 2$, their proof is based on the theorem of Abhyankar-Moh [AM] and Suzuki [Su], and the cancellation theorem of Abhyankar-Heinzer-Eakin [AHE]. Hence, the result is valid not only for $\mathbf{C}$, but also for any field of characteristic zero. By Proposition 3.1 below, the statement holds for a more general class of commutative rings. Recall that a commutative ring k with identity is said to be *steadfast* if the following condition holds for any commutative ring A containing k (cf. [H]): *If the polynomial rings $k[x_1, \ldots, x_n]$ and $A[x_2, \ldots, x_n]$ are k-isomorphic for some $n \geq 1$, then $k[x_1]$ and A are k-isomorphic.*

The following remark is due to Amartya K. Dutta who answered the author's question on his visit to Indian Statistical Institute in 2013 (see also [BD]).

Proposition 3.1. *Let k be a commutative ring with identity such that $k[x_1]$ is steadfast. If $f \in k[x_1, x_2]$ is a coordinate of $k[x_1, \ldots, x_n]$ for some $n \geq 3$, then f is a coordinate of $k[x_1, x_2]$.*

Proof. Put $A = k[x_1, x_2]$ and $k' = k[f]$. Then, there exist $f_2, \ldots, f_n \in k[x_1, \ldots, x_n]$ such that $A[x_3, \ldots, x_n] = k'[f_2, \ldots, f_n]$. Hence, the polynomial ring $k'[y_2, \ldots, y_n]$ over k' is k'-isomorphic to $A[x_3, \ldots, x_n]$ via the isomorphism defined by $y_i \mapsto f_i$ for each i. Since $k' \simeq k[x_1]$ is steadfast by assumption, it follows that $k'[y_2]$ and A are k'-isomorphic. Thus, we have $A = k'[g] = k[f, g]$ for some $g \in A$. Therefore, f is a coordinate of A. $\square$

For example, integrally closed domains are steadfast due to Asanuma [A] (see also [AHE] and [H]). If k is an integrally closed domain, then $k[x_1]$ is also an integrally closed domain. Hence, every stable coordinate of $k[x_1, x_2]$ is a coordinate of $k[x_1, x_2]$ by Proposition 3.1. On the other hand, Neena Gupta pointed out that Bhatwadekar-Dutta [BD, Example 4.1] constructed a "residual variable" of $k[x_1, x_2]$ which is not a coordinate of $k[x_1, x_2]$ when $k = \mathbf{Z}_{(2)}[2\sqrt{2}]$. Here, for a commutative noetherian ring k, the notion of residual variable of $k[x_1, x_2]$ is equivalent to the notion of stable coordinate of $k[x_1, x_2]$ (cf. [BD, Theorem A]). Therefore, not every stable coordinate of $k[x_1, x_2]$ is a coordinate of $k[x_1, x_2]$.

In the case of $n = 3$, Shpilrain-Yu used the cancellation theorem of Miyanishi-Sugie [MS] and Fujita [Fu], and the result of Kaliman [K] to show that every stable coordinate of $\mathbf{C}[\mathbf{x}]$ is a coordinate of $\mathbf{C}[\mathbf{x}]$. Neena Gupta

remarked that one can prove a similar statement over any field of characteristic zero by using the results of Sathaye [Sa] and Bass-Connell-Wright [BCW] instead of [K].

Next, we discuss stable $\mathbf{G}_{\mathrm{a}}$-invariants of $k[\mathbf{x}]$. In what follows, we assume that k is a field of characteristic zero. Then, for a k-subalgebra A of $k[\mathbf{x}]$, we have $A = k[\mathbf{x}]^{\mathbf{G}_{\mathrm{a}}}$ for some $\mathbf{G}_{\mathrm{a}}$-action on $k[\mathbf{x}]$ if and only if $A = \ker(D)$ for some locally nilpotent k-derivation of $k[\mathbf{x}]$. The following result [Fr2, Corollary 5.40] (see also [Fr1]) is a corollary to [Fr2, Theorem 5.37] which is due to Daigle and Freudenburg.

Proposition 3.2. *Assume that $n \geq 2$. Let D be a locally nilpotent k-derivation of $k[\mathbf{x}]$ with $k[x_1, x_2] \cap \ker(D) \neq k$. Then, we have either $D(x_1) = D(x_2) = 0$, or $k[x_1, x_2] \cap \ker(D) \subset k[g] \subset \ker D$ for some coordinate g of $k[x_1, x_2]$.*

In the situation of Proposition 3.2, we have $k[x_1, x_2] \cap \ker(D) = k[g]$ unless $D(x_1) = D(x_2) = 0$, since $k[x_1, x_2] \cap \ker(D)$ and $k[g]$ are both algebraically closed in $k[x_1, x_2]$ and of transcendence degree one over k. Hence, for any $\mathbf{G}_{\mathrm{a}}$-action on $k[\mathbf{x}]$ with $k[x_1, x_2] \cap k[\mathbf{x}]^{\mathbf{G}_{\mathrm{a}}}$ not equal to k or $k[x_1, x_2]$, there exists a coordinate g of $k[x_1, x_2]$ such that $k[x_1, x_2] \cap k[\mathbf{x}]^{\mathbf{G}_{\mathrm{a}}} = k[g]$.

Now, we show that every stable $\mathbf{G}_{\mathrm{a}}$-invariant of $k[x_1, x_2]$ is an invariant for a nontrivial $\mathbf{G}_{\mathrm{a}}$-action on $k[x_1, x_2]$. Take any stable $\mathbf{G}_{\mathrm{a}}$-invariant f of $k[x_1, x_2]$ not belonging to k. By definition, there exists a $\mathbf{G}_{\mathrm{a}}$-action on $k[\mathbf{x}]$ such that $k[\mathbf{x}]^{\mathbf{G}_{\mathrm{a}}}$ contains f, but does not contain $k[x_1, x_2]$. Then, $A := k[x_1, x_2] \cap k[\mathbf{x}]^{\mathbf{G}_{\mathrm{a}}}$ is not equal to k or $k[x_1, x_2]$. Hence, we have $A = k[g]$ for some coordinate g of $k[x_1, x_2]$ as remarked. As mentioned after Theorem 1.4, $k[g] = k[x_1, x_2]^{\mathbf{G}_{\mathrm{a}}}$ holds for some nontrivial $\mathbf{G}_{\mathrm{a}}$-action on $k[x_1, x_2]$. Since f is an element of $A = k[g] = k[x_1, x_2]^{\mathbf{G}_{\mathrm{a}}}$, we know that f is an invariant for this action of $\mathbf{G}_{\mathrm{a}}$.

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Department of Mathematics and Information Sciences
Tokyo Metropolitan University
1-1 Minami-Osawa, Hachioji
Tokyo 192-0397, Japan
kuroda@tmu.ac.jp